Path integral fermionization of two-dimensional sigma -models related to the homogeneous ones

1991 J. Phys. A: Math. Gen. 24 L395

(http://iopscience.iop.org/0305-4470/24/8/001)

J. Phys. A: Math. Gen. 24 (1991) L395–L398. Printed in the UK

LETTER TO THE EDITOR

Path integral fermionization of two-dimensional σ-models related to the homogeneous ones

A V Bratchikov

Krasnodar Polytechnic Institute, 2 Moskovskaya Street, Krasnodar 350072, USSR

Received 29 January 1991

Abstract. The generalization of the Polyakov–Wiegmann method of fermionization to a two-dimensional σ-model related to the one on an arbitrary homogeneous space G/H is presented. The generating functional for correlation functions of the model is expressed in the fermion path integral variables. The corresponding fermionic model is a constrained gauge model of a four-fermion interaction.

The two-dimensional σ-model on the homogeneous space G/H of a simple compact Lie group G is defined by the action [1]

$$S(U)=\int \mathrm{d}^2x(\tfrac{1}{2}\lambda_{ij}L^i_\mu L^{j\mu}) \tag{1a}$$

where L^i_μ, $i=1,\ldots,\dim(\mathrm{G/H})$, are given by the decomposition $U^{-1}\partial_\mu U=L^i_\mu\Gamma_i+L^a_\mu\Gamma_a$, $U=U(x)\in \mathrm{G}$, and $\lambda=(\lambda_{ij})$ is an inversible H-invariant matrix:

$$\sum_{k=1}^{\dim(\mathrm{G/H})}(\lambda_{ki}F^j_{ak}+\lambda_{kj}F^i_{ak})=0$$
$$F^i_{ak}=\mathrm{tr}([\Gamma_a,\Gamma_k]\Gamma_i). \tag{1b}$$

The Γ_i are generators of G in an irreducible unitary representation, satisfying the conditions $\mathrm{tr}(\Gamma_i\Gamma_a)=0$, where Γ_a, $a=\dim(\mathrm{G/H})+1,\ldots,\dim \mathrm{G}$, are generators of a group $\mathrm{H}\subset\mathrm{G}$ in the same representation. The action (1) is invariant under the global left-hand and gauge right-hand transformations

$$U(x)\to gU(x)h^{-1}x \tag{2}$$

where $g\in\mathrm{G}$, $h(x)\in\mathrm{H}$.

Using the Polyakov and Wiegmann method [2], one can show that the model (1) in the particular case $H=1$, $\lambda\sim 1$, constrained as [3]

$$U(x^+,0)=U(0,x^-)=1 \tag{3}$$

is equivalent to the infinite flavour limit of the G-invariant Thirring model. In this article we consider the action (1) in the general case. Its fermionization encounters the same difficulty as that of the principal chiral model [3, 4]. However, adding the constraints (3) one can apply the approach of [2, 3].

0305-4470/91/080395+04$03.50

This new model turns out to be equivalent to the constrained fermionic one which is defined by the action

$$S(\psi, \bar{\psi}, A^a_\mu) = \int d^2x(i\bar{\psi}_f \not{\partial} \psi^f + \lambda^{ij} J_{i\mu} J^\mu_j + iA^a_\mu J^\mu_a) \tag{4}$$

where $J_{\alpha\mu} = \bar{\psi}_f \gamma_\mu \Gamma_\alpha \psi^f$, $f = 1, \ldots, N$, $\alpha = 1, \ldots, \dim G$. The matrix $\lambda^{-1} = (\lambda^{ij})$ is the inverse of λ, and A^a_μ are Lagrange multipliers. A generalization of the method of [2] enables us to express in the fermion path integral variables not only the vacuum-to-vacuum functional integral of the model (1), (3), but also the generating functional for correlation functions.

Following Polyakov and Wiegmann [2], we first give arguments explaining the connection between (1) and (4) on a qualitative level. In order to eliminate the quartic interaction, it is convenient to introduce an auxiliary field A^i_μ and replace the action (4) by the new one

$$S(\psi, \bar{\psi}, A_\mu) = \int d^2x(i\bar{\psi}_f \not{D} \psi^f + \tfrac{1}{2}\lambda_{ij} A^i_\mu A^{j\mu}) \tag{5}$$

where $D_\mu = \partial_\mu + A_\mu$, $A_\mu = A^\alpha_\mu \Gamma_\alpha$, $\alpha = 1, \ldots, \dim G$. Integrating over fermions, we have

$$S(A_\mu) = \int d^2x(\tfrac{1}{2}\lambda_{ij} A^i_\mu A^{j\mu}) - iN \operatorname{tr} \log \not{D}. \tag{6}$$

Due to the gauge invariance, $\operatorname{tr}\log \not{D}$ depends effectively on the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Therefore at $N = \infty$, $F_{\mu\nu} = 0$ [2] and the field A_μ become a pure gauge. If it is represented as $A_\mu = U^{-1}\partial_\mu U$, we get the action (1).

To obtain the proof of the equivalence, let us consider the generating functional for correlation functions $Z_B(K)$ of the bosonic theory. The constrained model (1), (3) remains invariant under the gauge right-hand transformation (2) with $h(x^+, 0) = h(0, x^-) = I$, and we have

$$Z_B(K) = \int DU\, \delta_+[U]\delta_-[U]\delta[L_-] \exp\left(i\left(S(U) + \int d^2x\, K\phi(U)\right)\right) \tag{7}$$

where

$$\delta_+[U] = \prod_{x^+} \delta(U(x^+, 0) - I)$$

$$\delta_-[U] = \prod_{x^-} \delta(U(0, x^-) - 1)$$

$$\delta[L_-] = \prod_{x,a} \delta(L^a_-(x)).$$

The $L^a_- = L^a_0 - L^a_1$ are the gauge fixing functions, and $K = K(x)$ is an external source. The function $\phi(U)$ is supposed to be a gauge invariant.

It is easily verified [2] that $Z_B(K)$ (7) can be obtained at the $N \to \infty$ limit of the following functional integral

$$Z_F(K) = \int DU\, D\Omega\, \delta_+[U]\delta_-[\Omega U]\delta[L_-]$$
$$\times \exp\left(i\left(S(U) - iN\Gamma(\Omega) + \int d^2x(\tfrac{1}{2}\lambda_{ij} M^i_+ L^j_- + K\phi(U))\right)\right)$$

where M_+^i are defined by the decomposition

$$U^{-1}(\Omega^{-1}\partial_+\Omega)U = M_+^i\Gamma_i + M_+^a\Gamma_a \qquad \partial_\pm = \partial_0 \pm \partial_1$$

and

$$\Gamma(\Omega) = \frac{1}{8\pi}\int d^2x\, \mathrm{tr}(\partial_\mu\Omega\partial_\mu\Omega^{-1}) + \frac{1}{24\pi}\int d^3y\, \varepsilon^{ijk}\, \mathrm{tr}(\Omega^{-1}\partial_i\Omega\Omega^{-1}\partial_j\Omega\Omega^{-1}\partial_k\Omega).$$

Here, the last term is the Wess-Zumino one.

Let us change the integration variables from (Ω, U) to (A_+, A_-):

$$\Omega = VU^{-1} \qquad V = P\exp\left(\int_0^{x^+} A_+(\xi, x^-)\, d\xi\right)$$
$$U = P\exp\left(\int_0^{x^-} A_-(x^+, \xi)\, d\xi\right). \tag{8}$$

The corresponding Jacobian is a constant [3] and we have†

$$Z_F(K) = \int DA_\mu\, \delta[A_-]\exp\left(i\left(S(A_\mu) + \int d^2x\, K\phi(U)\right)\right) \tag{9}$$

where $S(A_\mu)$ is given by (6). To prove (8) and (9) equivalence we use the fact that $\mathrm{tr}\log \not{D} = -i\Gamma(\Omega)$ for $A_+ = \Omega^{-1}\partial_+\Omega$, $A_- = 0$ [2].

Introducing fermion variables one can rewrite $Z_F(K)$ as

$$Z_F(K) = \int D\psi\, D\bar\psi\, DA_\mu\delta[A_-]\exp\left(i\left(S(\psi, \bar\psi, A_\mu) + \int d^2x\, K\phi(U)\right)\right)$$

where $S(\psi, \bar\psi, A_\mu)$ is given by (5). Notice, that this functional integral contains the δ-function

$$iJ_{i-} + \lambda_{ij}A_-^j = 0 \tag{10}$$

which appears after the integration over A_+^i. Integrating then over A_-^i, we get at last

$$Z_F(K) = \int D\psi\, D\bar\psi\, DA_\mu^H \exp\left(i\left(S(\psi, \bar\psi, A_\mu^a) + \int d^2x\, K\phi(U)\right)\right)$$

where $DA_\mu^H = \Pi_{x,a}\, dA_\mu^a(x)\, \delta(A_-^a(x))$, and $S(\psi, \bar\psi, A_\mu^a)$ is the fermionic action (4). In the source term A_-^i is replaced by $-i\lambda^{ij}J_{j-}$ and, due to the condition $A_-^a = 0$, it depends effectively on the fermion variables only.

The formulae (8) and (10) enable us to express the boson variables as (non-local) functions of the fermion ones. For example, for the (constrained) model of the principal chiral field we have

$$U = P\exp\left(-i\int_0^{x^-} d\xi(\bar\psi_f\gamma_-\Gamma_\alpha\psi^f)\lambda^{\alpha\beta}\Gamma_\beta\right)$$

where α, $\beta = 1, \ldots, \dim G$, $f = 1, \ldots, \infty$. Similar fermionization relations can be obtained for the σ-models on the other symmetric [5] and homogeneous spaces.

† We do not keep track of constant factors in front of $Z_F(K)$.

The symmetric models are the most interesting among the σ-models (1). Almost all of them are solvable. Their S-matrices can be found explicitly [6–10] by the method based on the S-matrix factorization. What can by said about the corresponding fermionic models with a finite flavour number N? In [2, 11, 12] the Bethe Anzatz solutions of the N-flavour Thirring models corresponding to the principal chiral models were obtained. The N-flavour fermionic models (4), corresponding to the other symmetric ones, can be supposed solvable as well. If this were the case, we should get a new class of solvable two-dimensional modes.

I wish to thank I V Tyutin for useful discussions.

References

[1] Boulware D G and Brown L S 1982 *Ann. Phys., NY* **138** 392
[2] Polyakov A and Wiegmann P B 1983 *Phys. Lett.* **131B** 121
[3] Destri C and de Vega H J 1988 *Phys. Lett.* **208B** 255
[4] Destri C and de Vega H J 1988 *Phys. Lett.* **201B** 245
[5] Brezin E, Hikami S and Zinn-Justin J 1980 *Nucl. Phys.* B **165** 528
[6] Zamolodchikov A B and Zamolodchikov Al B 1979 *Ann. Phys., NY* **120** 253
[7] Abdalla E, Abdalla M C B and Lima-Santos A 1984 *Phys. Lett.* **140B** 71
[8] Wiegmann P 1984 *Phys. Lett.* **142B** 173
[9] Abdalla M C B 1985 *Phys. Lett.* **164B** 71
[10] Abdalla E, Abdalla M C B and Forger M 1988 *Nucl. Phys.* B **297** 374
[11] Wiegmann P B 1984 *Phys. Lett.* **141B** 217
[12] Ogievetsky E, Reshetikhin N and Wiegmann P 1987 *Nucl. Phys.* B **280** 45